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ABOUT THE RATIO OF THE SIZE OF A MAXIMUM ANTICHAIN TO THE SIZE OF A MAXIMUM LEVEL IN FINITE PARTIALLY ORDERED SETS

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Let P be a finite partially ordered set. The length $l(x)$ of an element x of P is defined by the maximal number of elements, which lie in a chain with x at the top, reduced by one. Let $w(P)$ $(d(P))$ be the maximal number of elements of P which have the same length (which form an antichain). Further let $P^n := \underbrace{P\times\ldots\times P}_{n\text{-times}}$. The numbers $r_k := \max\limits_{P:|P|=k} \frac{d(P)}{w(P)}$ and $s_k := \max\limits_{P:|P|=k} \lim\limits_{n\to\infty} \frac{d(P^n)}{w(P^n)}$ as well as all partially ordered sets for which these maxima are attained are determined.

1. Introduction

Let P be a finite partially ordered set (poset). Let $C=(c_0<\ldots<c_a)$ be a chain in P and let $l(C):=|C|-1=a$. We say, C has length $l(C)$. To each element u of P we may associate a unique number $l(u)$ (the length of u) which is equal to the maximal length of a chain from any minimal element in P to u. Let $l(P):=\max\limits_{u\in P} l(u)$. All elements of length i form the i-th level $N_i(P)$. Let $W_i(P):=|N_i(P)|$ and $w(P):=\max\limits_{i\in\{0,\ldots,l(P)\}} W_i(P)$. Further let $d(P)$ be the size of a maximum antichain in P, i.e. the maximal number of pairwise incomparable elements in P. It is well-known that each level of P is an antichain. Consequently, $\frac{d(P)}{w(P)} \geqq 1$ for each poset P. The posets for which $d(P)=w(P)$ are called Sperner posets. In many papers Sperner posets are investigated, and several sufficient conditions are known that a poset or a direct product of posets is a Sperner poset; see, for instance, [3], [5], [6] and [9]. (The direct product $P\times Q$ of two posets P and Q is defined on the Cartesian product of the sets P and Q as follows: $(u_1, v_1) \underset{P\times Q}{\leqq} (u_2, v_2)$ iff $u_1 \underset{P}{\leqq} u_2$ and $v_1 \underset{Q}{\leqq} v_2$.) Let

$$r_k := \max_{P:|P|=k} \frac{d(P)}{w(P)} \quad\text{and}\quad s_k := \max_{P:|P|=k} \lim_{n\to\infty} \frac{d(P^n)}{w(P^n)},$$

where $P^n := \underbrace{P\times\ldots\times P}_{n\text{-times}}$. (In the proof of Theorem 2 it is proved that s_k and, especially, the corresponding limit always exists.)

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In our paper we will determine r_k and s_k. Further all posets P with $|P|=k$ for which $\frac{d(P)}{w(P)}=r_k$ or $\lim_{n\to\infty}\frac{d(P^n)}{w(P^n)}=s_k$ will be determined. We call them anti Sperner posets (a.S.p.) or limit anti Sperner posets (l.a.S.p.), respectively, since they can be regarded as extremal posets in contrary to the Sperner case. Since in general the values $w(P)$ or $w(P^n)$ can easily be determined or asymptotically estimated $(n\to\infty)$, respectively, we thus obtain "best" estimates for the size of a maximum antichain $(w(P)\leqq d(P)\leqq r_k w(P),\ w(P^n)\leqq d(P^n)\lesssim s_k w(P^n))$. In all what follows we suppose that $|P|\geqq 3$ (otherwise $\frac{d(P^n)}{w(P^n)}=1$, $n=1,2,\dots$, and all posets with $|P|<3$ are a.S.p. and l.a.S.p.).

2. Determination of r_k

Theorem 1. $r_k=\frac{1}{2}\left[\frac{k+2}{2}\right]$. *($[x]$ is the largest integer not greater than x.) If $|P|=$ $=k>3$ and k is odd (even), then there are $\frac{k-1}{2}+2^{\frac{k-1}{2}}$ (one) a.S.p. If $|P|=$ $=k=3$, we have 5 a.S.p.*

Proof. Obviously, all posets with $|P|=3$ are a.S.p., and there are 5 posets containing 3 elements. Let $|P|=k>3$. We put $p:=l(P)$. Then there exists a chain $(c_0<\dots<c_p)$ such that $l(c_i)=i\,(i=0,\dots,p)$. Evidently, $d(P)\leqq(k-p-1)+1=k-p$.

Case 1. $w(P)=1$.

Then $\frac{d(P)}{w(P)}=1<\frac{1}{2}\left[\frac{k+2}{2}\right]$.

Case 2. $w(P)=2$.

Then $p+1\geqq\left]\frac{k}{2}\right[$, where $]x[$ is the least integer not less than x. We have

$$\frac{d(P)}{w(P)}\leqq\frac{k-p}{w(P)}\leqq\frac{1}{2}\left(k+1-\left]\frac{k}{2}\right[\right)=\frac{1}{2}\left[\frac{k+2}{2}\right].$$

Equality holds iff $p+1=\left]\frac{k}{2}\right[$ and $d(P)=k-p$. From this one can conclude (further details are omitted): If k is even, then there is only one a.S.p. which is given in Figure 1 by its Hasse diagram. If k is odd, there are $\frac{k-1}{2}+2^{(k-1/2)}$ a.S.p. which can be obtained from the a.S.p. of $k+1$ elements as follows: Omit an arbitrary element of $\{d_0,\dots,d_p\}\left(p=\frac{k-1}{2}\right)$ and, if d_p is omitted, add as one wants the relation $d_i<c_p$ $(i\in\{0,\dots,p-1\})$.

Case 3. $w(P)=3$.

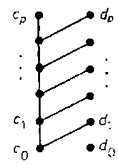

Fig. 1

Then $p+1 \geqq \left]\frac{k}{3}\right[$. Hence

$$\frac{d(P)}{w(P)} \leqq \frac{k-p}{w(P)} \leqq \frac{2k+3}{9} < \frac{1}{2}\left[\frac{k+2}{2}\right].$$

Case 4. $w(P) \geqq 4$.

Then $\frac{d(P)}{w(P)} \leqq \frac{k}{4} < \frac{1}{2}\left[\frac{k+2}{2}\right]$. ∎

3. Determination of s_k

Let

$$q_k := \begin{cases} \sqrt{\frac{k+2}{4}}, & \text{if } k \text{ is even,} \\ \sqrt{\frac{k+2+\frac{1}{k}}{4}}, & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 2. $q_k = s_k$. *There are only* 1 *or* 2 *l.a.S.p. if k is odd or even, respectively.*

Proof. If P is an antichain, then $\frac{d(P^n)}{w(P^n)} = 1$, $n=1, 2, \dots$, and P is not a l.a.S.p. In all what follows let P be not an antichain and $|P|=k$.

In order to formulate a result of V. B. Alekseev [1] we need the following definition. A *representation* of a poset P is a mapping $z\colon P \to \mathbf{R}$ such that $z(x)-z(y) \geqq 1$ if $x > y$. Let $N_c^z := \{x \in P;\, z(x)=c\}$. To each representation z of P one can associate a discrete random variable ξ^z as follows:

$$P(\xi^z = c) = \frac{|N_c^z|}{k}.$$

Let $E(\xi^z)$ and $V^2(\xi^z)$ be the expected value and variance of ξ^z, respectively. Now let $D(P) := \inf V^2(\xi^z)$, where the infimum is extended over all representations z of P. In [1], Lemma 1 it is proved that there is always a so-called optimal representation z^*

of P such that $V^2(\xi^{z*})=D(P)$. We need Theorem 2 of [1], (for related results see [2] and [4]) which reads

(1) $$d(P^n) \sim \frac{1}{\sqrt{2\pi D(P)}}\cdot\frac{k^n}{\sqrt{n}} \quad \text{as } n \to \infty.$$

From the proof of Theorem 1 in [7] one can easily derive that

(2) $$w(P^n) \sim \frac{1}{\sqrt{2\pi L(P)}}\cdot\frac{k^n}{\sqrt{n}} \quad \text{as } n \to \infty,$$

where $L(P):=V^2(\lambda)$, and λ is the discrete random variable for which

$$P(\lambda = i) = \frac{W_i(P)}{k}.$$

We conclude from (1) and (2) that

$$\lim_{n\to\infty}\frac{d(P^n)}{w(P^n)} = \sqrt{\frac{L(P)}{D(P)}}.$$

Consequently, we must prove that $\frac{L(P)}{D(P)} \leqq q_k^2$ (in the end of the proof of Theorem 2 we show that there are actually posets for which equality holds). In the following we will estimate $L(P)$ and $D(P)$ under the supposition that $p:=l(P)$ and $t:=|N_p(P)|$ are fixed values. But at first we need the following.

Lemma 1. *Let ζ and ζ' be discrete random variables such that $P(\zeta=i)=a_i/k$, $P(\zeta'=i)=b_i/k$ $(i=0, \dots, p)$, where a_i and b_i are integers $(i=0, \dots, p)$ with $\sum_{i=0}^{p} a_i = \sum_{i=0}^{p} b_i = k$. If $j \leqq E(\zeta)$ and*

$$b_i = \begin{cases} a_i+1, & \text{if } i=j-1, \\ a_i-1, & \text{if } i=j, \\ a_i, & \text{otherwise} \end{cases}$$

or if $j \geqq E(\zeta)$ and

$$b_i = \begin{cases} a_i-1, & \text{if } i=j, \\ a_i+1, & \text{if } i=j+1, \\ a_i, & \text{otherwise} \end{cases}$$

then $V^2(\zeta') \geqq V^2(\zeta)$. ∎

This lemma can be proved by a short calculation. We say, ζ' can be obtained from ζ by a *shifting step*.

Lemma 2. *a)* $L(P) \leqq \frac{(p-1)^2}{4} + \frac{p}{6k}(-p^2+3p-2+6t).$

b) If in addition $k \leqq 1+p+\frac{2p}{p-1}t$, then

$$L(P) \leqq \frac{1}{k}\left(\frac{(p-1)p(2p-1)}{6}+tp^2\right) - \frac{1}{k^2}\left(\frac{p(p-1)}{2}+tp\right)^2.$$

Proof. If we apply many times a shifting step starting with λ such that $P(v=p)=\frac{t}{k}$ for every random variable v which appears in the shifting procedure, we obtain that $L(P)=V^2(\lambda)$ is not greater than $V^2(\mu)$, where

$$P(\mu=0)=\frac{k-p-x-t+1}{k},$$
$$P(\mu=i)=\frac{1}{k}\ (i=1,\dots,p-2),$$
$$P(\mu=p-1)=\frac{x+1}{k},$$
$$P(\mu=p)=\frac{t}{k},$$

(see Figure 2).

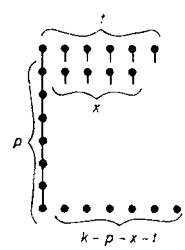

Fig. 2

If we consider $V^2(\mu)$ as a function of x then the maximum is attained on $x=\frac{k-p}{2}-\frac{p}{p-1}t$ and its value is that given in a). Since we estimate $L(P)$ in the case $p=l(P)$ we have at least one element in the level $N_{p-1}(P)$. Hence, we may suppose that $x\geqq 0$. If $k\leqq 1+p+\frac{2p}{p-1}t$, then the function $V^2(\mu)$ depending on integers $x\geqq 0$ attains its maximum on $x=0$. The value of this maximum is that given in b). ∎

Lemma 3. $D(P)\geqq\frac{p}{12(p+t)k}(p^3+4tp^2+6tp-p+2t)$.

Proof. Let z^* be an optimal representation of P. Then let c_p be such an element of $N_p(P)$ for which $z^*(c_p)=\min_{u\in N_p(P)} z^*(u)$. Obviously, there is a chain $C=(c_0<\dots<c_p)$. We have

$$(3)\qquad \begin{aligned} z^*(c_i)-z^*(c_{i-1}) &\geqq 1 \quad (i=1,\dots,p) \quad \text{and} \\ z^*(u) &\geqq z^*(c_p) \quad (u\in N_p(P)). \end{aligned}$$

If $x:=E(\xi^{z^*})$, then

$$(4)\qquad D(P)=V^2(\xi^{z^*})\geqq\sum_{i=0}^{p-1}\frac{1}{k}(z^*(c_i)-x)^2+\sum_{u\in N_p(P)}\frac{1}{k}(z^*(u)-x)^2.$$

If $x > z^*(c_p)$, then because of (3) and (4)

$$D(P) \geq \sum_{i=0}^{p-1} \frac{1}{k}(p-i)^2. \tag{5}$$

If $x \leq z^*(c_p)$, then let j be the smallest integer for which $z^*(c_j) \geq x$. Now we have because of (3) and (4)

$$D(P) \geq \sum_{i=-j}^{p-j-1} \frac{1}{k}(z^*(c_j)+i-x)^2 + \frac{t}{k}(z^*(c_j)+p-j-x)^2 = S(x+j-z^*(c_j)),$$

where $S(y) := \sum_{i=0}^{p-1} \frac{1}{k}(y-i)^2 + \frac{t}{k}(y-p)^2$. Inequality (5) can now be written as $D(P) \geq S(p)$. If we minimize $S(y)$ as a function of y we obtain the value given in the lemma. ▮

We continue with the proof of Theorem 2.

Case 1. $p \geq 4$.

Because of Lemma 2 and 3 we have

$$\frac{L(P)}{D(P)} \geq F(p, t, k),$$

where

$$F(p, t, k) := \frac{\frac{(p-1)^2}{4} + \frac{p}{6k}(-p^2+3p-2+6t)}{\frac{p}{12(p+t)k}(p^3+4tp^2+6tp-p+2t)}.$$

We shall prove that

$$F(p, t, k) < \frac{k+2}{4}, \tag{6}$$

i.e. no poset P with $l(P) \geq 4$ is a l.a.S.p. Inequality (6) is equivalent to $A_1 < B_1 \cdot k$, where

$$A_1 := 48pt^2 + (-16p^3+60p^2-20p)t + (-10p^4+24p^3-14p^2) \text{ and}$$

$$B_1 := (4p^3-6p^2+26p-12)t + (p^4-12p^3+23p^2-12p).$$

Because of $t \geq 1$ it is $B_1 \geq p^2((p-4)^2+1) > 0$. Moreover we have $k \geq p+t$ since there are t elements of P in $N_p(P)$ and $c_0, \ldots, c_{p-1}$ are further p elements of P. Thus, it is sufficient to prove that $A_1 < B_1 \cdot (p+t)$ which is equivalent to

$$A_2 t^2 + B_2 t + C_2 > 0, \tag{7}$$

where $A_2 := 4p^3 - 6p^2 - 22p - 12$, $B_2 := 5p^4 - 2p^3 - 11p^2 - 4p$, $C_2 := p^5 - 2p^4 - p^3 + 2p^2$. Since A_2, B_2, and $C_2 > 0$ (for $p \geq 4$) and $t \geq 1$ the inequality (7) is true.

Case 2. $p = 3$.

From Lemmas 2 and 3 we obtain

$$L(P) \leqq 1+\frac{1}{2k}(6t-2),$$

$$L(P) \leqq \frac{5+9t}{k}-\left(\frac{3+3t}{k}\right)^2 \quad \text{if} \quad k \leqq 4+3t, \quad \text{and}$$

$$D(P) \geqq \frac{14t+6}{(t+3)k}.$$

We will prove that there is not a l.a.S.p.

Case 2.1. $k>4+3t$.

It is sufficient to prove that

$$\frac{1+\frac{1}{2k}(6t-2)}{\frac{14t+6}{(t+3)k}} < \frac{k+2}{4}$$

which is equivalent to $(10t-6)k>12t^2+4t-24$. The last inequality is true since $k>4+3t$ and $(10t-6)(4+3t)>12t^2+4t-24$.

Case 2.2. $k \leqq 4+3t$.

It is sufficient to prove that

$$\frac{\frac{5+9t}{k}-\left(\frac{3+3t}{k}\right)^2}{\frac{14t+6}{(t+3)k}} < \frac{k+2}{4}$$

which is equivalent to

(8) $$A_3k^2+B_3k+C_3>0,$$

where $A_3:=\frac{7}{2}t+\frac{3}{2}$, $B_3:=-9t^2-25t-12$, and $C_3:=(9t^2+18t+9)(t+3)$. Since $4A_3C_3>B_3^2$ if $t \geqq 1$, inequality (8) is true.

Case 3. $p=2$.

This case can be treated analogously to Case 2.

Case 4. $p=1$.

We have $L(P)=\frac{t}{k}\left(1-\frac{t}{k}\right) \leqq \frac{1}{4}$. If there are elements u_1, v_1, u_2, v_2 of P such that $u_1<v_1$ and $u_2<v_2$ and if $x:=E(\xi^{z^*})$, then it holds because of $z^*(v_i)-z^*(u_i) \geqq 1$, $i=1,2$,

$$D(P) \geqq \frac{1}{k}\left((z^*(u_1)-x)^2+(z^*(v_1)-x)^2+(z^*(u_2)-x)^2+(z^*(v_2)-x)^2\right) \geqq$$

$$\geqq \frac{1}{k}.$$

Hence, $\frac{L(P)}{D(P)} \leqq \frac{k}{4}$, and such posets are not l.a.S.p.

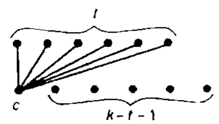

Fig. 3

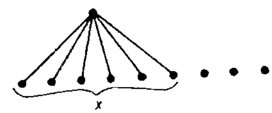

Fig. 4

There remain only posets of types given in Figures 3 and 4.

Consider first posets P of the first kind. Obviously, an optimal representation z^* is given by

$$z^*(u) := \begin{cases} 0, & \text{if } u = c, \\ \dfrac{t}{t+1}, & \text{if } u \in N_0(P) \setminus \{c\}, \\ 1, & \text{if } u \in N_1(P), \end{cases}$$

and it is $D(P) = \dfrac{t}{k(t+1)}$. Hence $\dfrac{L(P)}{D(P)} = \dfrac{(t+1)(k-t)}{k} \leqq q_k^2$, and equality only holds if

$$t = \begin{cases} \dfrac{k}{2} \text{ or } \dfrac{k}{2} - 1, & \text{if } k \text{ is even}, \\ \dfrac{k-1}{2}, & \text{if } k \text{ is odd}. \end{cases} \tag{9}$$

Finally consider posets P of the second kind. We may suppose $x \geqq 2$ since the case $x=1$ was settled above with $t=1$. Obviously, $L(P) = \dfrac{1}{k}\left(1 - \dfrac{1}{k}\right)$ and $D(P) = \dfrac{1}{k} \dfrac{x}{x+1} \geqq \dfrac{2}{3} \dfrac{1}{k}$. Hence,

$$\frac{L(P)}{D(P)} \leqq \frac{3}{2}\left(1 - \frac{1}{k}\right) < \frac{k+2}{4} \leqq q_k^2.$$

Consequently, only posets given in Figure 3 for which (9) is satisfied are l.a.S.p. ∎

4. Concluding remarks

Theorem 2 can be generalized in the following way. Let $\{S_n\}$ be a sequence of posets such that $|S_n| \leqq k$ $(n=1, 2, \ldots)$. We suppose that there are infinitely many posets of $\{S_n\}$ which are not antichains. We collect them in the subsequence $\{S_{n_i}\}$ $(i=1, 2, \ldots)$. Let $P_n := S_1 \times \ldots \times S_n$.

Theorem 3. $\limsup\limits_{n \to \infty} \dfrac{d(P_n)}{w(P_n)} \leqq q_k$.

Proof. At first we mention that $\frac{d(P\times A)}{w(P\times A)}=\frac{d(P)}{w(P)}$ if A is an antichain. Hence,

$$\limsup_{n\to\infty}\frac{d(P_n)}{w(P_n)}=\limsup_{i\to\infty}\frac{d(Q_i)}{w(Q_i)}, \tag{10}$$

where $Q_i:=S_{n_1}\times\ldots\times S_{n_i}$. Let $k_i:=|S_{n_i}|$. In [4] it is proved that

$$d(Q_i)\sim\frac{k_1\ldots k_i}{\sqrt{2\pi(D(S_{n_1})+\ldots+D(S_{n_i}))}}\quad\text{as}\quad i\to\infty.$$

Analogously to (2) one can derive from Petrov's limit theorem for k-sequences of independent random variables (see [8], p. 189) that

$$w(Q_i)\sim\frac{k_1\ldots k_i}{\sqrt{2\pi(L(S_n)+\ldots+L(S_{n_i}))}}\quad\text{as}\quad i\to\infty$$

($D(S_{n_j})$ and $L(S_{n_j})$ are defined as in the beginning of 3.). We conclude

$$\frac{d(Q_i)}{w(Q_i)}=\sqrt{\frac{\sum_{j=0}^{i}L(S_{n_j})}{\sum_{j=0}^{i}D(S_{n_j})}\frac{(1+o_1(1))}{(1+o_2(1))}}. \tag{11}$$

From the proof of Theorem 2 we obtain

$$\frac{L(S_{n_j})}{D(S_{n_j})}\leqq q_{k_j}^2\leqq q_k^2\,(j=1,\ldots,i).$$

Now the inequality given in Theorem 3 can be derived from (10), (11), and (12). ∎

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